XI. *On certain Definite Integrals occurring in Spherical Harmonic Analysis and on the Expansion, in Series, of the Potentials of the Ellipsoid and the Ellipse.*

By W. D. Niven, *M.A., Fellow of Trinity College, Cambridge.*

Communicated by J. W. L. Glaisher, *M.A., F.R.S.*

Received April 3,—Read May 1, 1879.

§ 1. The object of this paper is to explain a method of dealing with a class of results, some of which are of frequent occurrence and some of considerable importance, in the solution of Laplace's equation in series.

The fundamental theorem on which the method depends is expressed by the integration of

$$\iint e^{\alpha x+\beta y+\gamma z} dS$$

over a sphere whose centre is the origin of coordinates and whose radius is R. This sphere we shall call the sphere of reference.

By change of axes the above integral takes the form

$$2\pi R\int_{-R}^{R} e^{x\sqrt{(\alpha^2+\beta^2+\gamma^2)}}dx$$

and its value is

$$2\pi R\frac{e^{R\sqrt{(\alpha^2+\beta^2+\gamma^2)}}-e^{-R\sqrt{(\alpha^2+\beta^2+\gamma^2)}}}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or, in series,

$$4\pi R^2\left\{1+\frac{R^2}{3!}(\alpha^2+\beta^2+\gamma^2)+\ldots+\frac{R^{2i}}{2i+1!}(\alpha^2+\beta^2+\gamma^2)^i+\ldots\right\} \quad . \quad . \quad . \quad (2)$$

It follows that if V be any function whose value may be expressed for all points within the sphere by a convergent series, according to Taylor's theorem, or by the symbolical form

$$e^{x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}}V_0$$

then the integral $\iint V dS$ taken over the surface of the sphere is

$$4\pi R^2\sum_0^\infty\frac{R^{2i}}{2i+1!}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)^i V_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The corresponding theorem in two dimensions, which we will also employ, is this :—The integral

$$\int e^{\alpha x+\beta y}ds$$

taken round the periphery of a circle of reference, is

$$2\pi\mathrm{R}\sum_0^\infty\frac{\mathrm{R}^{2i}}{2^{2i}i!\,i!}(\alpha^2+\beta^2)^i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence it follows that

$$\int \mathrm{V}ds,$$

taken round the same periphery, is

$$2\pi\mathrm{R}\sum_0^\infty\frac{\mathrm{R}^{2i}}{2^{2i}i!\,i!}\left(\frac{d^2}{dx}+\frac{d^2}{dy}\right)^i\mathrm{V}_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The theorems expressed by (3) and (5) are employed below in a variety of cases connected with spherical harmonics. It is therefore necessary to specify at the outset the notation to be used.

We denote, according to custom, the colatitude and longitude of a point referred to the axes of reference by θ and ϕ: $\cos\theta$ and $\sin\theta$ by μ and ν respectively. We denote the LEGENDRE'S coefficient or zonal harmonic of the i^{th} degree by the symbol P_i so that P_i is defined by the equation

$$i!\,\mathrm{P}_i=(-1)^i r^{i+1}\frac{d^i}{dz}\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

According to CLERK MAXWELL'S theory of poles the general expression for the harmonic of the i^{th} degree is given by the equation

$$i!\,\mathrm{Y}_i=(-1)^i r^{i+1}\frac{d^i}{dh_1dh_2\ldots dh_i}\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $\frac{d}{dh}$ means differentiation with regard to an axis h.

In the harmonic (7) the i poles are the points where the i axes of differentiation cut the sphere of reference. In the harmonic (6) the i poles all coincide with the point where the axis of z cuts the sphere. If the i poles coincide at any other point the harmonic will be denoted by Q_i.

In the tesseral and sectorial system we have ventured to depart from the usual notations by denoting them as follows :—

$$i!\,(i,\sigma)=(-1)^i r^{i+1}\left(\frac{d\sigma}{d\xi}+\frac{d^\sigma}{d\eta}\right)\frac{d^{i-\sigma}}{dz}\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$i!\,[i,\sigma]=(-1)^i r^{i+1}\jmath\left(\frac{d^\sigma}{d\xi}-\frac{d^\sigma}{d\eta}\right)\frac{d^{i-\sigma}}{dz}\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $j=\sqrt{-1}$, $\xi=x+jy$, $\eta=x-jy$. It will be observed that (8) is not in accordance with the general definition (7), for if we put $\sigma=0$ we get $(i, o)=2P_i$.

As regards the name of Spherical Harmonic introduced by THOMSON and TAIT, it seems desirable that the names of LEGENDRE and LAPLACE should be in some way retained in connexion with the different kinds of harmonics with which they are associated. From the point of view of the former writers there is perhaps some difficulty in doing this. We shall however continue to give the name of LAPLACE'S coefficient to what THOMSON and TAIT call the Biaxal Harmonic or the formula which expresses Q_i in terms of the harmonics referred to fixed axes of reference. With the definitions given above, this is

$$Q_i=P_iP_i'+, \text{\&c.}, +\frac{2^{2\sigma-1}i!\,i!}{i+\sigma!\,i-\sigma!}\{(i, \sigma)(i, \sigma)'+[i, \sigma][i, \sigma]'\}+ \quad . \quad . \quad (10)$$

in which any accented function is derived from the corresponding unaccented function by the substitution of the coordinates of the pole of Q_i in place of the running coordinates.

The following are, in the first instance, the integrals to which the method of this paper will be applied :—

(1) $\iint \mu^m P_n dS$

(2) $\iint (1-\mu^2)^m P_{2n} dS$

(3) $\iint (1-\mu^2)^{m-\frac{1}{2}} P_{2n} dS$

(4) $\iint P_p P_q P_r dS$

(5) $\iint (p, \alpha)(q, \beta)(r, \alpha+\beta) dS$

$\iint (p, \alpha)[q, \beta][r, \alpha+\beta] dS$, &c.

(6) $\iint P_p P_q P_r P_s dS$

Of these cases the first three require no special comment. The fourth case was for the first time solved by Professor J. C. ADAMS, in the Proceedings of the Royal Society, vol. xxvii., pp. 63–71, although the result had been independently found and published by Mr. FERRERS in his 'Treatise on Spherical Harmonics.'

Professor ADAMS, as well as Mr. FERRERS, made the discovery of the value of (4) by an inductive process, and Mr. TODHUNTER showed in a subsequent number of the Proceedings how the proof could be thrown into a compact form. The present writer was, however, convinced that the integral could be found deductively according to the method described above, and the simple character of the result led him to hope that the same method might be applied in cases of greater complexity. Accordingly cases (5) and (6) have been dealt with, but they have not been solved with the completeness of case (4), the results being practically expressed in the form of series.

The most interesting as well as the most important applications of the present method are connected with ellipsoids and ellipses. We have thereby obtained in series of harmonics the potentials of

(1) An ellipsoidal shell.

(2) A solid ellipsoid.

(3) An elliptic plate of uniform density.

(4) An electric current in an elliptic circuit.

Of these cases (2) leads to the approximate determination of the forces expressing the mutual action between two solid ellipsoids or uniformly magnetized ellipsoids, and (4) in certain cases to the forces between two currents in elliptic circuits, the particular case of circular circuits being completely determinate.

Similar results also hold for rectangular solids and circuits.

The case (2) derives additional interest from the fact that it engaged the attention of LAGRANGE, who obtained the expansion as far as the first four terms. The same four terms have also been worked out in a very interesting paper on the potential of an ellipsoid at an external point, by Colonel A. R. CLARKE ('Philosophical Magazine,' 1877, vol. ii., pp. 458–461).

In regard to case (3), Professor CAYLEY, in the 'Proceedings of the Mathematical Society' for 1875, has obtained the solution in the form of an integral, from which he derives interesting properties of the potential depending on certain particular positions of the attracted point. The expansion in harmonic series would seem, however, to be practically more useful in determining the mutual forces between two electrical circuits.

Throughout the following investigations, the method of treating Spherical Harmonics introduced by THOMSON and TAIT will be almost exclusively adopted.

$\iint \mu^m P_n dS$

§ 2. Taking a point A on the axis of z produced negatively at a distance a from the origin, let us consider the result of integrating

$$\iint \frac{z^m}{r} dS, \text{ or } \iint z^m u dS$$

over the surface of the sphere, where r is the distance of any point P from A, and u is the reciprocal of r.

If we denote by the suffix 1 the fact of the operator being upon z^m, and by the suffix 2 of its being upon u, then the operator

$$\left(\frac{d^2}{dx}+\frac{d^2}{dy}+\frac{d^2}{dz}\right)^i$$

upon the product, $z^m u$, may be written,

$$\left\{\frac{d^2}{dx_2}+\frac{d^2}{dy_2}+\left(\frac{d}{dz_2}+\frac{d}{dz_1}\right)^2\right\}^i$$

that is, in virtue of u satisfying LAPLACE'S equation,

$$\left(\frac{d^2}{dz_1}+2\frac{d}{dz_1}\frac{d}{dz_2}\right)^i$$

Referring now to the general result (3), and picking out the general term, we find

$$\frac{4\pi R^{2i+2}}{2i+1!}\frac{i!}{n!\,i-n!}2^n\frac{d^{2i-n}}{dz_1}\frac{d^n}{dz_2}$$

In order that this may lead to a value not zero, we must have

$$2i-n=m$$

or

$$2i=m+n$$

Substituting then this value of i, and differentiating $z^m u$, putting x, y, $z=0$ after the differentiations, we get

$$(-1)^n\frac{4\pi R^{m+n+2}}{m+n+1!}\frac{\frac{m+n}{2}!}{n!\frac{m-n}{2}!}2^n m!\,n!\frac{1}{a^{n+1}}$$

But u can be expanded in a series of zonal harmonics, viz. : it is

$$\frac{1}{a}-\frac{R}{a^2}P_1+\ldots+(-1)^n\frac{R^n}{a^{n+1}}P_n+\ldots$$

Substituting this expansion in $\iint z^m u dS$, and equating the coefficients of the different powers of the reciprocals of a to the values already found for them, we obtain finally, in the case where m and n are integers,

$$\iint\mu^m P_n dS=4\pi R^2\frac{2^n m!\frac{m+n}{2}!}{m+n+1!\frac{m-n}{2}!} \quad \ldots\ldots\ldots \quad (11)$$

It is obvious from the above proof that $m+n$ must be an even number, and that n must not be greater than m. In all other cases in which m and n are integers, the integral must be zero.

It is obvious the same method of solution as the foregoing will apply generally towards determining

$$\iint \frac{f(z)}{r}\, d\mathrm{S}$$

where $f(z)$ is any function whose differential coefficients are finite at the origin. In fact, we find

$$\iint f(z)\mathrm{P}_n d\mathrm{S} = \pi(2\mathrm{R})^{n+2}\frac{n!}{2n+1!}\left[\frac{d^n}{dz}+\frac{\mathrm{R}^2}{2(2n+3)^4}\frac{d^{n+2}}{dz}+\frac{\mathrm{R}^4}{2.4(2n+3)(2n+5)}\frac{d^{n+4}}{dz}+\&\mathrm{c.}\right]f(z). \quad (12)$$

where, after the differentiations are performed, z is to be put equal to zero.

A result practically equivalent to the above is given by HEINE ("Kugelfunctionen," second edition, p. 76), and applied to various cases where the functions can be expanded in simple series of zonal harmonics. The functions are

$$(1+k\mu^2)^{-(n+\frac{3}{2})},\ (1-\mu^2)^{\frac{1}{2}},\ \sin^{-1}\mu,\ (1-\mu^2)^{-\frac{1}{2}}$$

Similar functions are also to be found expanded in TODHUNTER'S "Functions of LAPLACE, LAME, and BESSEL," §§ 48, 145–147. We shall now discuss more general cases of some of these functions.

$\iint(1-\mu^2)^m\mathrm{P}_{2n}d\mathrm{S}$

§ 3. The case now indicated will be discussed only for integral values of m and n. Let us consider the integral

$$\iint\frac{(x^2+y^2)^m}{r}d\mathrm{S}, \text{ or } \iint(x^2+y^2)^m u\, d\mathrm{S}$$

If we use the suffix 1 in any operator when it is upon $(x^2+y^2)^m$, and 2 when it is upon u, we have as the general term operating,

$$\frac{4\pi\mathrm{R}^{2i+2}}{2i+1!}\left(\left(\frac{d}{dx_1}+\frac{d}{dx_2}\right)^2+\left(\frac{d}{dy_1}+\frac{d}{dy_2}\right)^2+\frac{d^2}{dz_2}\right)^i$$

or, since

$$\frac{d}{dx_1}+\frac{d}{dx_2}=\frac{d}{d\xi_1}+\frac{d}{d\eta_1}+\frac{d}{d\xi_2}+\frac{d}{d\eta_2}$$

$$\frac{d}{dy_1}+\frac{d}{dy_2}=j\left(\frac{d}{d\xi_1}-\frac{d}{d\eta_1}+\frac{d}{d\xi_2}-\frac{d}{d\eta_2}\right)$$

the general term above is

$$\frac{4\pi\mathrm{R}^{2i+2}}{2i+1!}\left(4\left(\frac{d}{d\xi_1}+\frac{d}{d\xi_2}\right)\left(\frac{d}{d\eta_1}+\frac{d}{d\eta_2}\right)+\frac{d^2}{dz_2}\right)^i$$

that is, in virtue of u satisfying LAPLACE'S equation,

$$\frac{4\pi R^{2i+2}}{2i+1!}4^i\left(\frac{d}{d\xi_1}\frac{d}{d\eta_1}+\frac{d}{d\xi_1}\frac{d}{d\eta_2}+\frac{d}{d\xi_2}\frac{d}{d\eta_1}\right)^i$$

The general term of the operator just found is

$$\frac{4\pi R^{2i+2}}{2i+1!}4^i\frac{i!}{u!\,i-u!}\left(\frac{d^2}{d\xi_1 d\eta_1}\right)^u\left(\frac{d}{d\xi_1}\frac{d}{d\eta_2}+\frac{d}{d\xi_2}\frac{d}{d\eta_1}\right)^{i-u}$$

It is obvious that $i-u$ must be even, otherwise the differentiated expressions could not but vanish when $x=0$, $y=0$, $z=0$. We must thus have

$$i-u=2n$$

$$i+u=2m$$

Substituting these values in the general operator, and retaining only the middle term of the last factor, which alone in that factor will produce a result differing from zero, we get, as the expression of the effective operator,

$$\frac{4\pi R^{2i+2}}{2m+2n+1!}4^i\frac{m+n!}{m-n!\,n!\,n!}\left(\frac{d}{d\xi_1}\frac{d}{d\eta_1}\right)^m\left(\frac{d}{d\xi_2}\frac{d}{d\eta_2}\right)^n$$

of which the last factor may be replaced by

$$\left(-\frac{1}{4}\frac{d^2}{dz_2}\right)^n$$

And now performing the operations indicated, we find as the result

$$(-1)^n\frac{4\pi R^{2i+2}}{2m+2n+1!}\,4^m\,\frac{m+n!\,m!\,m!\,2n!}{m-n!\,n!\,n!}\,\frac{1}{a^{2n+1}}$$

Since $x^2+y^2=R^2(1-\mu^2)=R^2\nu^2$, and the term involving P_{2n} in the expansion of u in zonal harmonics is

$$\frac{R^{2n}}{a^{2n+1}}P_{2n}$$

we have, finally,

$$\iint(1-\mu^2)^m P_{2n}dS=4\pi R^2\frac{(-1)^n 4^m}{2m+2n+1!}\,\frac{m+n!\,m!\,m!\,2n!}{m-n!\,n!\,n!} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If $\phi(\nu^2)$, any function of ν^2, be expanded in the form

$$A_0+A_1\nu^2+\ldots+A_{n+p}\nu^{2n+2p}+\ldots$$

then it may be shown that

$$\int_0^1 \phi(\nu^2)\mathrm{P}_{2n}d\mu$$

$$=(-1)^n\frac{2n!}{1.3.5\ldots(4n+1)}\left(\mathrm{A}_n+\frac{(n+1)^2}{4n+3}\,2\mathrm{A}_{n+1}+\frac{(n+1)^2(n+2)^2}{(4n+3)(4n+5)1.2}\,2^2\mathrm{A}_{n+1}+\&\mathrm{c.}\right). \quad (14)$$

In certain cases this series admits of summation. For example, let us examine under what circumstances $\iint\mu^{-k}\mathrm{P}_{2n}d\mathrm{S}$ admits of being evaluated

$$\mu^{-k}=(1-\nu^2)^{-\frac{k}{2}}=\Sigma\frac{\frac{k}{2}\left(\frac{k}{2}+1\right)\ldots\left(\frac{k}{2}+n-1\right)}{n!}\nu^{2n}$$

Hence

$$\mathrm{A}_{n+1}=\frac{\frac{k}{2}+n}{n+1}\,\mathrm{A}_n$$

$$\mathrm{A}_{n+2}=\frac{\left(\frac{k}{2}+n\right)\left(\frac{k}{2}+n+1\right)}{(n+1)(n+2)}\mathrm{A}_n,\ \&\mathrm{c.}$$

If we now write

$$\alpha=\frac{k}{2}+n$$

$$\beta=n+1$$

$$\gamma=2n+\frac{3}{2}$$

we find for the integral $\iint\mu^{-k}\mathrm{P}_{2n}d\mathrm{S}$, the expression

$$(-1)^n\frac{k(k+2)(k+4)\ldots(k+2n-2)}{(2n+1)(2n+3)(4n+1)}\Sigma$$

where

$$\Sigma=1+\frac{\alpha\beta}{\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)1.2}+\ldots$$

$$=\frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\alpha)}{}^{*}$$

$$=\frac{\Gamma\left(2n+\frac{3}{2}\right)\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+1+\frac{1-k}{2}\right)}$$

$$=\frac{\left(n+\frac{1}{2}\right)\ldots\left(2n+\frac{1}{2}\right)}{\frac{1-k}{2}\ldots\left(\frac{1-k}{2}+n\right)}$$

* BERTRAND, 'Calcul. Intégral,' 1870, pp. 495–6.

The value of the integral under discussion is accordingly

$$(-1)^n\frac{k(k+2)\ldots(k+2n-2)}{(1-k)(3-k)\ldots(2n+1-k)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and the above reasoning shows that k must be less than 1 : it may, however, have any negative value.

$$\iint(1-\mu^2)^{m-\frac{1}{2}}\mathrm{P}_{2n}d\mathrm{S}$$

---

§ 4. To evaluate the integral now indicated, m and n being integers, we must modify our previous solution, for otherwise we should be led to infinite values of the differential coefficients at the origin.

Let us consider the integral

$$\iint(\xi\eta)^{m-\frac{1}{2}}\mathrm{V}d\mathrm{S}$$

If V is a function which is symmetrical round the axis of z we may write this

$$2\pi\int\rho^{2m}\mathrm{V}ds$$

where ds is an element of arc of any section of the sphere of reference containing the axis of z, and $\rho^2+z^2=\mathrm{R}^2$.

Now the expression (5) gives the form of solution in this case. If we put

$$\mathrm{V}=\frac{1}{\pi}\int_0^\pi(z+j\rho\cos\psi)^{2n}d\psi$$

we have, in fact,

$$\iint(1-\mu^2)^{m-\frac{1}{2}}\mathrm{P}_{2n}d\mathrm{S}=\frac{4\pi\mathrm{R}^2}{2^{2m+2n}\,\underline{m+n}!\,\underline{m+n}!}\times\int_0^\pi\left(\frac{d^2}{d\rho}+\frac{d^2}{dz}\right)^{m+n}\rho^{2m}(z+j\rho\cos\psi)^{2n}d\psi$$

Expanding the quantities under the integral sign, and performing the differentiations and integrations, we obtain

$$\frac{2\pi^2\mathrm{R}^2 2m!2n!}{2^{2m+2n}\,\underline{m+n}!\,m!\,n!}\mathrm{S}$$

where

$$\mathrm{S}=1-\frac{n(m+\frac{1}{2})}{1.2}+\frac{n(n-1)(m+\frac{1}{2})(m+\frac{1}{2}+1)}{1.2\qquad\qquad 1.2}-\&\text{c.}$$

$$=\frac{(n-m-\frac{1}{2})(n-m-\frac{1}{2}-1)\ldots(-m-\frac{1}{2}+1)}{1.2\ldots n}$$

First let $n>m$, then in S we have $n-m$ positive and m negative factors, and in that case S becomes

$$(-1)^m \frac{\underline{2n-2m}!\,\underline{2m}!}{2^{2n}\,\underline{n-m}!\,m!\,n!}$$

Next, let $n<m$, then S becomes

$$(-1)^n \frac{\underline{2m}!\,\underline{m-n}!}{2^{2n}\,\underline{2m-2n}!\,m!\,n!}$$

Substituting these values of S in the expression for the integral we find

(1) if $n>m$

$$\frac{(-1)^m 2\pi^2 R^2 \underline{2m}!\ \underline{2n}!\ \underline{2n}!\ \underline{2n-2m}!}{2^{2m+4n}\,\underline{n+m}!\ \underline{n-m}!\ m!\ m!\ n!\ n!} \quad \ldots \quad (16)$$

(2) If $n<m$

$$\frac{(-1)^n 2\pi^2 R^2 \underline{2m}!\ \underline{2m}!\ \underline{2n}!\ \underline{m-n}!}{2^{2m+4n}\,\underline{m+n}!\ \underline{2m-2n}!\ m!\ m!\ n!\ n!} \quad \ldots \quad (17)$$

$\iint Q_i \frac{dP_j}{d\mu} dS, \iint Q_i \frac{d^2P_j}{d\mu^2} dS$

§ 5. The results for the two integrals now stated are, if α be the colatitude of the pole of Q_i,

$$4\pi R^2 P_i(\cos\alpha) \quad \ldots \quad (18)$$

$$2\pi R^2 (j-i)(j+i+1) P_i(\cos\alpha) \quad \ldots \quad (19)$$

where, in the second integral, $j-i$ must be a positive even integer.

GENERAL THEOREMS IN DIFFERENTIATION.

§ 6. Before we proceed with the remaining cases, it will be convenient to state and prove various theorems in differentiation.*

Theorem i. The general operator (7) upon the reciprocal of r may be made to operate, instead, upon a homogeneous function in x, y, z of the i^{th} degree; this effect being expressed by the relation

$$(-1)^i \frac{d^i}{dh_1 dh_2 \ldots dh_i} \frac{1}{r} = \frac{1}{r^{i+1}} \frac{d^i}{dh_1 dh_2 \ldots dh_i} (r^i Q_i) \quad \ldots \quad (20)$$

where the pole of Q_i is in the direction of r.

* Theorems i. and ii. were given in a paper by the author in the 'Messenger of Mathematics,' No. 73, 1877. On account of the brevity of the proofs, and in order to secure completeness, it is thought best to reproduce them here.

Theorem ii. relates to the differentiation of the reciprocal of $r^{2\sigma+1}$ and may be stated thus:

$$(-1)^{i-\sigma}\frac{d^{i-\sigma}}{dh_1dh_2\ldots dh_{i-\sigma}}\frac{1}{r^{2\sigma+1}}=\frac{2^\sigma\sigma!}{2\sigma!}\frac{d^{i-\sigma}}{dh_1dh_2\ldots dh_{i-\sigma}}\left(r^{i-\sigma}\frac{d^\sigma Q_i}{d\mu^\sigma}\right) \quad . \quad . \quad . \quad (21)$$

Theorem iii. If $f\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)$ be any homogeneous operator of the i^{th} degree operating on a homogeneous expression $\phi\ (x, y, z)$ of the i^{th} degree, then

$$f\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)\phi(x, y, z)=\phi\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)f\,(x, y, z) \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This theorem is obviously true, and though stated for only three quantities, x, y, z, is true for any number.

Theorems i. and ii. are proved as follows: Let $OP=r$, $QP=\rho$ and $O\hat{P}Q=\pi-\theta$, where Q is any point near to P. Then

$$\frac{1}{OQ}=\frac{1}{r}-\frac{\rho}{r^2}Q_1+\ldots+(-1)^i\frac{\rho^i}{r^{i+1}}Q_i+\&c.$$

Keeping r fixed let us perform the operation (7) on the two sides of this equation and then put $\rho=0$, in which case OQ becomes r. It is to be observed that with a homogeneous operator like (7), it is immaterial what point is taken for origin of coordinates. If then we take P for origin when we are dealing with the right hand side of the above expansion, we see that the first i terms will disappear by repeated differentiations, and the terms beyond the $i+1^{th}$ disappear in consequence of the zero value of ρ. The theorem stated therefore follows, the substitution of r for ρ making no difference in a result from which x, y, z are finally made to disappear.

Theorem ii. may be proved in the same way, if instead of the expansion for the reciprocal of OQ, as above, we take the expansion for the reciprocal $OQ^{2\sigma+1}$, found by differentiating the above series σ times with regard to μ.

DIFFERENT FORMS OF TESSERAL AND SECTORIAL HARMONICS.

§ 7. In several of the following investigations frequent use will be made of different forms of the zonal and tesseral harmonics. With the view of classifying these various forms, and of bringing the various expressions for the tesseral and sectorial system into harmony with the corresponding expressions for the zonal, a proof is here given which will be a direct illustration of the foregoing theorems.

The most important forms of the zonal harmonic P_i are

$$\frac{1}{2^i i!}\frac{d^i}{d\mu}(\mu^2-1)^i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

$$\frac{2i}{2^i i!\, i!}\left(\mu^i - \frac{i(i-1)}{2(2i-1)}\mu^{i-2} + \ldots\right) \quad \ldots \ldots \quad \text{(B)}$$

$$\frac{1}{\pi}\int_0^\pi (\mu + j\nu\cos\psi)^i d\psi, \quad \frac{1}{\pi}\int_0^\pi \frac{d\psi}{(\mu + j\nu\cos\psi)^{i+1}} \quad \ldots \ldots \quad \text{(C)}$$

$$\mu^i - \frac{i(i-1)}{4}\mu^{i-2}\nu^2 + \ldots \ldots \quad \text{(D)}$$

Now by definition

$$(i, \sigma) = (-1)^i \frac{r^{i+1}}{i!}\frac{d^{i-\sigma}}{dz}\left(\frac{d^\sigma}{d\xi} + \frac{d^\sigma}{d\eta}\right)\frac{1}{\sqrt{z^2 + \xi\eta}}$$

$$= (-1)^{i+\sigma}\frac{2\sigma!}{2^{2\sigma} i!\, \sigma!} r^{i+1}(\xi^\sigma + \eta^\sigma)\frac{d^{i-\sigma}}{dz}\ \frac{1}{r^{2\sigma+1}}$$

By Theorem ii. this is

$$\frac{1}{2^\sigma i!} 2\cos\sigma\phi\nu^\sigma \frac{d^{i-\sigma}}{dz}\left(r^{i-\sigma}\frac{d^\sigma Q_i}{d\mu^\sigma}\right)$$

We observe that, in this result, the axis of the harmonic Q_i is the line OP, as explained in the proof of Theorem i. We must therefore change the axes of coordinates, making the new axis of z coincide with OP.

If $r^{i-\sigma}\frac{d^\sigma P_i}{d\mu^\sigma}$, when expanded, becomes

$$Az^{i-\sigma} - Bz^{i-\sigma-2}\rho^2 + \ldots$$

we have to consider the effect of operating upon this with

$$\left(\cos\alpha\frac{d}{dz} - \sin\alpha\frac{d}{d\rho}\right)^{i-\sigma}$$

There will result

$$i-\sigma!\ (A\cos^{i-\sigma}\alpha - B\cos^{i-\sigma-2}\alpha\sin^2\alpha + \ldots)$$

that is

$$i-\sigma!\ \frac{d^\sigma P_i}{d\mu^\sigma}$$

where μ is now the cosine of the angle between OP and the axis of z. Hence we obtain the first and second forms of the harmonic corresponding to the forms (A) and (B) of the zonal harmonic, viz. :—

$$(i, \sigma) = \frac{i-\sigma!}{2^\sigma i!} 2\cos\sigma\phi\nu^\sigma\frac{d^\sigma P_i}{d\mu^\sigma} \quad \ldots \ldots \quad (a)$$

$$= \frac{2\, i!}{2^{i+\sigma} i!\, i!} 2\cos\sigma\phi\nu^\sigma\left(\mu^{i-\sigma} - \frac{(i-\sigma)(i-\sigma-1)}{2(2i-1)}\mu^{i-\sigma-2} + \ldots\right) \quad \ldots \quad (b)$$

Let us now consider the definite integral

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{p\cos^2\phi+q\sin^2\phi}=\frac{\pi}{2}\frac{1}{\sqrt{pq}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

This result will still be true if for p, q we substitute the unreal quantities $c+re^{\theta j}$, $c+re^{-\theta j}$, as may indeed be shown by direct integration.

Let us differentiate both sides of the equation (α), σ times with regard to p and σ times with regard to q. We find

$$2\sigma!\int_0^{\frac{\pi}{2}} \frac{\sin^{2\sigma}\phi\cos^{2\sigma}\phi}{(p\cos^2\phi+q\sin^2\phi)^{2\sigma+1}}d\phi=\frac{\pi}{2}\left\{\frac{1.3\ldots(2\sigma-1)}{2^\sigma}\right\}^2\frac{1}{(pq)^{\sigma+\frac{1}{2}}}$$

Making the above substitutions for p, q we get

$$\frac{2^\sigma\sigma!}{1.3\ldots 2\sigma-1}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin^{2\sigma}2\phi}{\{c+r(e^{\theta j}\cos^2\phi+e^{-\theta j}\sin^2\phi)\}^{2\sigma+1}}d\phi=\left(\frac{1}{c^2+2cr\mu+r^2}\right)^{\frac{2\sigma+1}{2}}$$

$$=\frac{1}{1.3\ldots(2\sigma-1)}\frac{1}{c^{2\sigma+1}}\left(\frac{d^\sigma P_\sigma}{d\mu^\sigma}+\ldots+(-1)^{i-\sigma}\left(\frac{r}{c}\right)^{i-\sigma}\frac{d^\sigma P_i}{d\mu^\sigma}+\ldots\right)$$

the last line being derived from its predecessor by σ differentiations with regard to μ.

And now expanding the expression under the integral sign in powers of r, and equating the coefficients of $r^{i-\sigma}$, we find

$$\frac{d^\sigma P_i}{d\mu^\sigma}=\frac{i+\sigma!\ \sigma!}{2\sigma!\ i-\sigma!}2^\sigma\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(e^{\theta j}\cos^2\phi+e^{-\theta j}\sin^2\phi)^{i-\sigma}\sin^{2\sigma}2\phi d\phi$$

On putting $2\phi=\psi$, and substituting the expression just found in equation (a), we finally obtain for (i, σ) one of the forms corresponding to C, viz. :—

$$\frac{i+\sigma!\ \sigma!}{2\sigma!\ i!}2\cos\sigma\phi\nu^\sigma\frac{1}{\pi}\int_0^\pi(\mu+j\nu\cos\psi)^{i-\sigma}\sin^{2\sigma}\psi d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

There is obviously another expression, corresponding to the second form of C, which is easily written. It is to be found by expanding, as above, in powers of the reciprocal of r.

The expansion of the binomial inside of the integral (c) and subsequent integration with regard to ψ give the form corresponding to D, viz. :—

$$\frac{i+\sigma!}{2^{2\sigma}i!\ \sigma!}2\cos\sigma\phi\nu^\sigma\left\{\mu^{i-\sigma}-\frac{(i-\sigma)(i-\sigma-1)}{4(\sigma+1)}\mu^{i-\sigma-2}\nu^2+\ldots\right\} \quad . \quad . \quad . \quad . \quad (d)$$

Integration of the Product of Two Harmonics and Proof of LAPLACE'S *Coefficient.*

§ 8. Let us take two points A and B at distances a and b from the origin, A being on the axis of z and B at an angular distance α from it.

Let P be any point on the sphere of reference, and let us consider the integral

$$\iint\frac{dS}{AP.BP}$$

By (3) we may write as the result of this

$$4\pi R^2\sum_0^\infty\frac{R^{2i}}{2i+1!}\left(\frac{d^2}{dx}+\frac{d^2}{dy}+\frac{d^2}{dz}\right)^i\frac{1}{AO.BO}$$

If we use the suffix 1 to denote differentiation upon AP only and 2 upon BP only, the above becomes by virtue of LAPLACE's equation

$$4\pi R^2\sum_0^\infty\frac{R^{2i}}{2i+1!}2^i\left(\frac{d}{dx_1}\frac{d}{dx_2}+\frac{d}{dy_1}\frac{d}{dy_2}+\frac{d}{dz_1}\frac{d}{dz_2}\right)^i\frac{1}{AO.BO}$$

or, what is the same thing,

$$4\pi R^2\sum_0^\infty\frac{R^{2i}}{2i+1!}2^i\left(2\frac{d}{d\xi_1}\frac{d}{d\eta_2}+2\frac{d}{d\xi_2}\frac{d}{d\eta_1}+\frac{d}{dz_1}\frac{d}{dz_2}\right)^i\frac{1}{AO.BO}$$

Now on consideration of Theorem i. we will see that in expanding the operator just found, the only terms which will lead to results not zero will be those which, so far as the operator with suffix 1 is concerned, are of the form

$$\frac{d^m}{d\xi_1}\frac{d^m}{d\eta_1}\frac{d^n}{dz_1}$$

and this by virtue of LAPLACE's operator

$$\frac{d^2}{dz_1}+4\frac{d}{d\xi_1}\frac{d}{d\eta_1}=0$$

is expressible in terms of $\frac{d}{dz_1}$ only. Hence we conclude that the only effective terms may be found directly as those not containing a power of v in the expansion of

$$4\pi R^2\sum_0^\infty\frac{R^{2i}}{2i+1!}\left(-v^2\frac{d^2}{dz_1}-\frac{1}{v^2}\frac{d^2}{dz_2}+2\frac{d}{dz_1}\frac{d}{dz_2}\right)^i\frac{1}{AO.BO}$$

i.e., in

$$4\pi R^2\sum_0^\infty(-1)^i\frac{R^{2i}}{2i+1!}\left(v\frac{d}{dz_1}-\frac{1}{v}\frac{d}{dz_2}\right)^{2i}\frac{1}{AO.BO}$$

i.e.,

$$4\pi R^2\sum\frac{R^{2i}}{2i+1!}\frac{2i!}{i!\,i!}\frac{d^i}{dz_1}\frac{d^i}{dz_2}\frac{1}{AO.BO}$$

Hence

$$\iint\frac{dS}{AP.BP}=4\pi R^2\sum\frac{R^{2i}}{2i+1}\frac{P_i(\cos\alpha)}{a^{i+1}b^{i+1}}$$

If now we expand the reciprocals of AP and BP in harmonics we arrive at the well-known theorems

$$\iint P_i Q_j dS = 0$$

$$\iint P_i Q_i dS = \frac{4\pi R^2}{2i+1} P_i(\cos\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The last result, when properly considered, can be made to yield the integral of any two harmonics. For it can be easily seen from the foregoing work that if ρ_1, ρ_2 are any two vectors drawn from the origin, then we have this very remarkable formula for Q_i

$$2i!\ Q_i = \rho_1^{i+1}\rho_2^{i+1}2^i\left(\frac{d}{dx_1}\frac{d}{dx_2}+\frac{d}{dy_1}\frac{d}{dy_2}+\frac{d}{dz_1}\frac{d}{dz_2}\right)^i\frac{1}{\rho_1\rho_2}$$

On account of the operator being an invariant we may suppose the axes to be any rectangular axes whatever.

By what has just been shown we may throw this into the form

$$2i!\ Q_i = \rho_1^{i+1}\rho_2^{i+1}(-1)^i\left(v\frac{d}{dz_1}-\frac{1}{v}\frac{d}{dz_2}\right)^{2i}\frac{1}{\rho_1\rho_2}$$

where v^2 stands for

$$\frac{d}{d\eta_2} \div \frac{d}{d\eta_1}$$

The general term of the value of $2i!\ Q_i$ just found is

$$\frac{\rho_1^{i+1}\rho_2^{i+1}(-1)^\sigma 2i!}{i-\sigma!\ i+\sigma!}\frac{v^{i-\sigma}}{v^{i+\sigma}}\quad\frac{d^{i-\sigma}}{dz_1}\quad\frac{d^{i+\sigma}}{dz_2}\quad\frac{1}{\rho_1\rho_2}$$

i.e.,

$$\frac{\rho_1^{i+1}\rho_2^{i+1}(-1)^\sigma 2i!}{i-\sigma!\ i+\sigma!}\ \frac{d^\sigma}{d\eta_1}\ \frac{d^{-\sigma}}{d\eta_2}\ \frac{d^{i-\sigma}}{dz_1}\ \frac{d^{i+\sigma}}{dz_2}\ \frac{1}{\rho_1\rho_2}$$

or

$$\frac{\rho_1^{i+1}\rho_2^{i+1}2i!}{i-\sigma!\ i+\sigma!}\ 2^{2\sigma}\frac{d^\sigma}{d\eta_1}\ \frac{d^\sigma}{d\xi_2}\ \frac{d^{i-\sigma}}{dz_1}\ \frac{d^{i-\sigma}}{dz_2}\ \frac{1}{\rho_1\rho_2}$$

It is obvious a term similar to this can be obtained with only ξ_1 put for ξ_2 and η_2 for η_1, by merely changing the sign of σ in the first of the last three expressions. Adding the two terms and recollecting the definitions of the harmonics we may write the result

$$\frac{2i!\ i!\ i!}{i-\sigma!\ i+\sigma!}\ 2^{2\sigma-1}\{(i,\sigma)(i,\sigma)'+[i,\sigma][i,\sigma]'\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Hence LAPLACE'S Coefficient as given in § 1.

Reverting now to equation (23), let us take two fixed poles, and let Q_i, Q_i' be expanded according to LAPLACE'S Formula; we shall then be at once led to the well-known surface integrals of two tesseral harmonics.

The discussion entered into in this article is necessary for the treatment of the product of three harmonics to which we now proceed.

It may be remarked that the expansion of the operator

$$\left(\frac{d}{dx_1}\frac{d}{dx_2}+\frac{d}{dy_1}\frac{d}{dy_2}+\frac{d}{dz_1}\frac{d}{dz_2}\right)^i\frac{1}{\rho_1\rho_2}$$

is, according to the above investigation,

$$\left\{2^{-i}\frac{2i!}{i!\,i!}\frac{d^i}{dz_1}\frac{d^i}{dz_2}+\ldots+2^{2\sigma-i}\frac{2i!}{i+\sigma!\,i-\sigma!}\left(\frac{d^\sigma}{d\xi_1}\frac{d^\sigma}{d\eta_2}+\frac{d^\sigma}{d\xi_2}\frac{d^\sigma}{d\eta_1}\right)\left(\frac{d}{dz_1}\frac{d}{dz_2}\right)^{i-\sigma}+\ldots\right\}\frac{1}{\rho_1\rho_2}$$

(See also THOMSON and TAIT, p. 157).

General Method of Dealing with the Integral $\iint Q_pQ_qQ_r\ldots dS$.

§ 9. In discussing the general method of evaluating this integral it will be convenient to confine ourselves to the case of three harmonics, though the first steps of the reasoning will apply to any number.

Let A, B, C be any three points whose distances from the origin are a, b, c, and let P be any point on the sphere of reference. Then by § 1

$$\iint\frac{dS}{\text{AP.BP.CP}}=4\pi\text{R}^2\Sigma\frac{\text{R}^{2i}}{2i+1!}\left(\frac{d^2}{dx}+\frac{d^2}{dy}+\frac{d^2}{dz}\right)^i\frac{1}{\text{AO.BO.CO}}$$

$$=4\pi\text{R}^2\Sigma\frac{\text{R}^{2i}}{2i+1!}\,2^i(v_{23}+v_{31}+v_{12})^i\frac{1}{\text{AO.BO.CO}}$$

where v_{23} stands for

$$\frac{d}{dx_2}\frac{d}{dx_3}+\frac{d}{dy_2}\frac{d}{dy_3}+\frac{d}{dz_2}\frac{d}{dz_3},\ \&\text{c.}$$

the suffixes having the same signification as in the previous article.

The general term of $2i+2$ dimensions in R may therefore be written

$$\frac{4\pi\text{R}^{2i+2}}{2i+1!}2^i\frac{i!}{\lambda!\,\mu!\,\nu!}\,v^\lambda{}_{23}v^\mu{}_{31}v^\nu{}_{12}\frac{1}{\text{AO.BO.CO}}$$

Now the general expansion of each of the operators in this expression was found in the previous article.

If therefore we adopt a somewhat more convenient notation we may expand $v_{23}{}^\lambda$, $v_{31}{}^\mu$, $v_{12}{}^\nu$ in a series of terms of which the type is

$$2^{2(l+m+n)-i}\frac{2\lambda!\,2\mu!\,2\nu!}{\lambda+l!\,\lambda-l!\,\mu+m!\,\mu-m!\,\nu+n!\,\nu-n!}$$

$$\times\left(\frac{d^l}{d\xi_2}\frac{d^l}{d\eta_3}+\frac{d^l}{d\xi_3}\frac{d^l}{d\eta_2}\right)\left(\frac{d^m}{d\xi_3}\frac{d^m}{d\eta_1}+\frac{d^m}{d\xi_1}\frac{d^m}{d\eta_3}\right)\left(\frac{d^n}{d\xi_1}\frac{d^n}{d\eta_2}+\frac{d^n}{d\xi_2}\frac{d^n}{d\eta_1}\right)$$

$$\times\frac{d^{\mu+\nu-m-n}}{dz_1}\frac{d^{\nu+\lambda-n-l}}{dz_2}\frac{d^{\lambda+\mu-l-m}}{dz_3}$$

where the values of l, m, n range between 0 and λ, μ, ν.

The general operator just found we shall denote by the symbol π (l, m, n).

§ 10. If we expand each of the reciprocals of AP, BP, CP in harmonics we shall arrive at the general term of $2i+2$ dimensions in R in a different form, viz.: it will be

$$\frac{\mathrm{R}^{2i}}{a^{p+1}b^{q+1}c^{r+1}}\iint \mathrm{Q}_p\mathrm{Q}_q\mathrm{Q}_r d\mathrm{S}$$

In order therefore that the general term as found by the work of the last article may correspond with this we must have

$$\mu+\nu=p$$
$$\nu+\lambda=q$$
$$\lambda+\mu=r$$

The quantities λ, μ, ν, as depending upon p, q, r, are therefore perfectly determinate, and the equation expressing the identity of the general terms may by Theorem i. be written

$$\iint \mathrm{Q}_p\mathrm{Q}_q\mathrm{Q}_r d\mathrm{S}=\frac{4\pi\mathrm{R}^2}{2i+1!}2^i\frac{i!}{\lambda!\,\mu!\,\nu!}\Sigma\Sigma\Sigma\pi(l,\,m,\,n)(\mathrm{Q}_p r_1^{\,p}\mathrm{Q}_q r_2^{\,q}\mathrm{Q}_r r_3^{\,r})\quad\ldots\quad(25)$$

On examining the equations determining λ, μ, ν, we see that in order to their being positive, $p+q+r$ must be an even number, and no one of the three quantities p, q, r, must be greater than the sum of the other two.

If these conditions are not satisfied, then $\iint \mathrm{Q}_p\mathrm{Q}_q\mathrm{Q}_r d\mathrm{S}=0$ for integral values of p, q, r, wherever the poles of the harmonics may be.

$\iint \mathrm{P}_p\mathrm{P}_q\mathrm{P}_r d\mathrm{S}$

§ 11. Let us now suppose that the three points A, B, C are in the axis of z. Then the harmonic Q_p becomes P_p, which we will suppose expressed by the series D. In like manner P_q and P_r may be similarly expressed. It follows that, in selecting from the general operator what terms are practically effective, and discarding those which are not, we need retain only such terms as are, for example, of the form

$$\frac{d^\sigma}{d\xi_1}\frac{d^\sigma}{d\eta_1}\frac{d^{p-2\sigma}}{dz_1}$$

Now the terms of this form are obviously got by putting $l=m=n$, $=\sigma$ say. In that case, when we omit inoperative terms, π becomes

$$2^{6\sigma-i+1}\frac{2\lambda!\,2\mu!\,2\nu!}{\lambda+\sigma!\,\lambda-\sigma!\,\mu+\sigma!\,\mu-\sigma!\,\nu+\sigma!\,\nu-\sigma!}$$

$$\times\left(\frac{d}{d\xi_1}\frac{d}{d\eta_1}\frac{d}{d\xi_2}\frac{d}{d\eta_2}\frac{d}{d\xi_3}\frac{d}{d\eta_3}\right)^\sigma\frac{d^{p-2\sigma}}{dz_1}\frac{d^{q-2\sigma}}{dz_2}\frac{d^{r-2\sigma}}{dz_3}$$

that is, in virtue of LAPLACE'S equation being satisfied,

$$(-1)^\sigma\frac{2}{2^i}\frac{2\lambda!\,2\mu!\,2\nu!}{\lambda+\sigma!\,\lambda-\sigma!\,\mu+\sigma!\,\mu-\sigma!\,\nu+\sigma!\,\nu-\sigma!}\frac{d^p}{dz_1}\frac{d^q}{dz_2}\frac{d^r}{dz_3}$$

Taking all possible terms of π, such as that just found, and performing the operation directed by them, we have for $\iint P_pP_qP_r dS$, the expression

$$\frac{4\pi R^2}{2i+1!}\frac{2\lambda!\,2\mu!\,2\nu!\,i!}{\lambda!\,\mu!\,\nu!}\left(\frac{p!\,q!\,r!}{(\lambda!\,\mu!\,\nu!)^2}+2\Sigma(-1)^\sigma\frac{p!\,q!\,r!}{\lambda+\sigma!\,\lambda-\sigma!\,\mu+\sigma!\,\mu-\sigma!\,\nu+\sigma!\,\nu-\sigma!}\right)$$

where σ ranges from 0 to the smallest of the quantities λ, μ, ν.

Now the series within brackets is clearly the coefficient of $x^{2\lambda}$, $y^{2\mu}$, $z^{2\nu}$, or of $x^{\mu+\nu}$, $y^{\nu+\lambda}$, $z^{\lambda+\mu}$, in

$$(-1)^i(y-z)^{\mu+\nu}(z-x)^{\nu+\lambda}(x-y)^{\lambda+\mu}$$

or, again, the same series is also obviously equal to

$$\frac{p!\,q!\,r!}{2\lambda!\,2\mu!\,2\nu!}K$$

where K is the coefficient of $x^{\mu+\nu}\,y^{\nu+\lambda}\,z^{\lambda+\mu}$ in

$$(-1)^i(y-z)^{2\lambda}(z-x)^{2\mu}(x-y)^{2\nu}$$

that is, in

$$(-1)^i x^{\mu+\nu}y^{\nu+\lambda}z^{\lambda+\mu}\left(\sqrt{\frac{y}{z}}-\sqrt{\frac{z}{y}}\right)^{2\lambda}\left(\sqrt{\frac{z}{x}}-\sqrt{\frac{x}{z}}\right)^{2\mu}\left(\sqrt{\frac{x}{y}}-\sqrt{\frac{y}{x}}\right)^{2\nu}$$

If we now write herein

$$z=xe^{2\phi j}=ye^{2\theta j}$$

we see that K is the term not involving the cosine of an angle of the value $m\theta+n\phi$ in the expansion in such cosines of W, where

$$W=(-1)^i\, 2^{2i} \sin^{2\lambda}\theta \sin^{2\mu}\phi \sin^{2\nu}(\theta-\phi)$$

so that

$$K=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi W d\theta d\phi$$

Now expanding $\sin^{2\nu}(\theta-\phi)$ in powers of sines and cosines by the binomial theorem, we observe that the odd terms in the expansion disappear on integration, so that the value of K is given by

$$\Sigma\frac{1}{\pi^2}\int_0^\pi\int_0^\pi(-1)^i\, 2^{2i}\frac{2\nu!}{2n!\; 2\nu-2n!}\sin^{2\lambda+2n}\theta\cos^{2\nu-2n}\theta\sin^{2\mu+2\nu-2n}\phi\cos^{2n}\phi\, d\theta\, d\phi$$

On integration and reduction this becomes

$$(-1)^i\, 2^{-2\nu}\frac{2\nu!}{\lambda+\nu!\;\mu+\nu!}\Sigma\frac{2\lambda+2n!\;2\mu+2\nu-2n!}{\lambda+n!\;\mu+\nu-n!\;\nu-n!\;n!}$$

that is,

$$(-1)^i\, 2^{-2\nu}\frac{2\nu!}{\lambda+\nu!\;\mu+\nu!}\;\frac{2\lambda!}{\lambda!}\;\frac{2\mu+2\nu!}{\mu+\nu!}\;\frac{1}{\nu!}S$$

where

$$S=1+\frac{(2\lambda+1)\nu}{2\mu+2\nu-1}+\frac{(2\lambda+1)(2\lambda+3)\nu(\nu-1)}{(2\mu+2\nu-1)(2\mu+2\nu-3)1.2}+\&c.$$

This hypergeometric series is capable of summation by a method similar to that given by BERTRAND, 'Calcul. Intégral,' 1870, pp. 495–496. If we put

$$\lambda+\tfrac{1}{2}=\alpha$$
$$\mu+\nu-\tfrac{1}{2}=\beta$$

S becomes

$$1+\frac{\alpha\nu}{\beta}+\frac{\alpha(\alpha+1)}{\beta(\beta-1)}\,\frac{\nu(\nu-1)}{1.2}+\;\&c.$$

$$=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\,\frac{\Gamma(\beta-n+1)}{\Gamma(\beta+1)}\,\frac{\nu(\nu-1)\ldots(\nu-n+1)}{n!}$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\Sigma\int_0^\infty\frac{\nu(\nu-1)\ldots\nu-n+1}{n!}\,\frac{y^{\alpha+n-1}}{(1+y)^{\alpha+\beta+1}}dy$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma\beta+1)}\int_0^\infty\frac{y^{\alpha-1}dy}{(1+y)^{\lambda+\mu+1}}$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\,\frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(\lambda+\mu+1)}$$

$$=2^{2\nu}\frac{i!}{\lambda+\mu!}\,\frac{2\mu!}{\mu!}\,\frac{\mu+\nu!}{2\mu+2\nu!}$$

On substituting this value for S in the value of K we find

$$K=(-1)^i\frac{i!\,2\lambda!\,2\mu!\,2\nu!}{p!\,q!\,r!\,\lambda!\,\mu!\,\nu!}$$

Finally, performing the operations with regard to the z's, we find

$$\iint P_pP_qP_r dS=\frac{4\pi R^2}{2i+1!}\,\frac{i!\,i!}{2i!}\,\frac{2\lambda!\,2\mu!\,2\nu!}{\lambda!\,\lambda!\,\mu!\,\mu!\,\nu!\,\nu!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

This agrees with the result obtained by Professor ADAMS.

§ 12. Before leaving this case we may remark that, so far as the foregoing proof is concerned, the poles of all the three harmonics are not necessarily coincident. The proof will hold when two are coincident and the third is at an angular distance, say α, from them. In those circumstances the value of $\iint P_pP_qQ_r dS$ will be the result found above multiplied by $P_r(\cos\alpha)$. The same is, of course, true of the integral $\iint Q_pQ_qP_r dS$, provided the two Q's have the same poles. Now let Q_p, Q_q be each expanded according to LAPLACE's Formula. Then we will have the curious result that any zonal harmonic of degree r can be expanded in products of harmonics, zonal and tesseral, of degrees p and q, the quantities p, q, r being, of course, subject to the foregoing restrictions. If q is less than p the number of terms in such expansion will be $2q+1$

$$\iint (p,\,\alpha)(q,\,\beta)(r,\,\alpha+\beta)dS$$

---

§ 13. Reverting to the general expression for the operator $\pi\ (l,\,m,\,n)$ given in § 8, let us leave out for an instant the numerical multipliers and the operators with regard to the z's and let us multiply out those with regard to ξ and η. Let us suppose l, m, n in descending order of magnitude, and for the sake of brevity, instead of writing the differential operators, let us write only the characteristic letter, *e.g.*, instead of $\frac{d}{d\xi_1}$ write ξ_1. We find

$$\xi_1{}^n\eta_1{}^n\xi_2{}^n\eta_2{}^n\xi_3{}^m\eta_3{}^m(\xi_1{}^{m-n}\eta_2{}^{l-n}\xi_3{}^{l-m}+\eta_1{}^{m-n}\xi_2{}^{l-n}\eta_3{}^{l-m})$$
$$+\xi_1{}^n\eta_1{}^n(\xi_1{}^{m-n}\xi_2{}^{l+n}\eta_3{}^{l+m}+\eta_1{}^{m-n}\eta_2{}^{l+n}\xi_3{}^{l+m})$$
$$+\xi_2{}^n\eta_2{}^n(\xi_1{}^{m+n}\xi_2{}^{l-n}\eta_3{}^{l+m}+\eta_1{}^{m+n}\eta_2{}^{l-n}\xi_3{}^{l+m})$$
$$+\xi_3{}^m\eta_3{}^m(\xi_1{}^{m+n}\eta_2{}^{l+n}\xi_3{}^{l-m}+\eta_1{}^{m+n}\xi_2{}^{l+n}\eta_3{}^{l-m})$$

Now since any product such as $\frac{d}{d\xi_1}\,\frac{d}{d\eta_1}$ can be replaced by $-\frac{1}{4}\,\frac{d^2}{dz_1}$, we see that each of the lines just written corresponds to the sum of tesseral harmonics, in such manner that if we suppose the differentiations performed by the whole of the operator, and if one of the resulting terms is

$$\frac{L}{a^{p+1}b^{q+1}c^{r+1}}(p,\ \alpha)_a(q,\ \beta)_b(r,\ \gamma)_c$$

where the suffixes indicate that the angular coordinates of A, B, C are substituted in the harmonics to which they are respectively attached, then we have in all cases one of the quantities α, β, γ equal to the sum of the other two. This we see from the expanded form of the operator just given, and it is a result which was to be expected, for supposing Q_p, Q_q, Q_r severally expanded according to LAPLACE'S Formula we should have in the surface integral a series of terms of which the type involves the integral

$$\int_0^{2\pi} \cos\alpha\phi \cos\beta\phi \cos\gamma\phi d\phi$$

and this integral vanishes unless the above conditions be complied with, the case where $\alpha+\beta+\gamma=0$ being of course out of the reckoning.

In accordance with the explanation just given we may now put

$$\left.\begin{array}{ccc} m-n=\alpha & \text{or} & m+n=\alpha \\ l+n=\beta & & l-n=\beta \\ & \text{and} \therefore\ l+m=\alpha+\beta & \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If then we substitute in π $(l,\ m,\ n)$ for l and m the values $\beta+n$ and $\alpha-n$, we see that n may range in numerical value between 0 and the least of the integers $\lambda-\beta$, $\mu-\alpha$, ν, and may be positive or negative. The operator π, in fact, becomes

$$(-1)^n 2^{2(\alpha+\beta)-i}\frac{2\lambda!\ 2\mu!\ 2\nu!}{\lambda+\beta+n!\ \lambda-\beta-n!\ \mu+\alpha-n!\ \mu-\alpha+n!\ \nu+n!\ \nu-n!}$$

$$\times\left(\frac{d^\alpha}{d\xi_1}\frac{d^\beta}{d\xi_2}\frac{d^{\alpha+\beta}}{d\eta_3}+\frac{d^\alpha}{d\eta_1}\frac{d^\beta}{d\eta_2}\frac{d^{\alpha+\beta}}{d\xi_2}\right)\times\frac{d^{\mu+\nu-\alpha}}{dz_1}\frac{d^{\nu+\lambda-\beta}}{dz_2}\frac{d^{\lambda+\mu-\gamma}}{dz_3}$$

We have therefore now to consider a series in which any term is the product of (1) an invariable power of 2, (2) an invariable operator, (3) a variable coefficient whose parameter is n. If we denote the first two of these factors by the symbol ϖ, then the sum of all such terms is ϖK, where

$$K=\frac{2\lambda!\ 2\mu!\ 2\nu!}{\mu+\nu-\alpha!\ \nu+\lambda-\beta!\ \lambda+\mu+\alpha+\beta!}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and W is the coefficient of $x^{2\lambda}y^{2\mu}z^{2\nu}$ in

$$(-1)^i(y-z)^{\mu+\nu-\alpha}(z-x)^{\nu+\lambda-\beta}(x-y)^{\lambda+\mu+\alpha+\beta}$$

that is, in

$$(-1)^i(y-z)^{p-\alpha}(z-x)^{q-\beta}(x-y)^{r+\alpha+\beta}$$

The expression upon which the operator ϖ has to operate may be found by expressing Q_p, &c., by LAPLACE'S Formula, and using the series D: it is as follows :—

$$\frac{2^{-1}p!}{\alpha!\,p-\alpha!}\{(\xi_1^{\alpha}+\eta_1^{\alpha})(p,\alpha)_a-j(\xi_1^{\alpha}-\eta_1^{\alpha})[p,\alpha]_a\}z_1^{p-\alpha}$$

$$\times\frac{2^{-1}q!}{\beta!\,q-\beta!}\{(\xi_2^{\beta}+\eta_2^{\beta})(q,\beta)_b-j(\xi_2^{\beta}-\eta_2^{\beta})[q,\beta]_b\}z_2^{q-\beta}$$

$$\times\frac{2^{-1}r!}{\alpha+\beta!\,r-\alpha-\beta!}\{(\xi_3^{\alpha+\beta}+\eta_3^{\alpha+\beta})(r,\alpha+\beta)_c-j(\xi_3^{\alpha+\beta}-\eta_3^{\alpha+\beta})[r,\alpha+\beta]_c\}z_3^{r-\alpha-\beta}$$

If we omit irrelevant terms this becomes

$$\frac{2^{-3}p!\,q!\,r!}{\alpha!\,\beta!\,\alpha+\beta!\,p-\alpha!\,q-\beta!\,r-\alpha-\beta}\times\{(p,\alpha)_a(q,\beta)_b(r,\alpha+\beta)_c$$

$$-(p,\alpha)[q,\beta][r,\alpha+\beta]_c-[p,\alpha]_a(q,\beta)_b[r,\alpha+\beta]_c-[p,\alpha]_a[q,\beta]_b(r,\alpha+\beta)_c\}$$

$$\times(\xi_1^{\alpha}\xi_2^{\beta}\eta_3^{\alpha+\beta}+\eta_1^{\alpha}\eta_2^{\beta}\xi_3^{\alpha+\beta})z_1^{p-\alpha}z_2^{q-\beta}z_3^{r-\alpha-\beta}$$

The result of the operation is accordingly

$$2^{2(\alpha+\beta)-i-2}p!\,q!\,r!\,\mathrm{K}\{(p,\alpha)_a(q,\beta)_b(r,\alpha+\beta)_c-\&c.\ldots\}$$

Turning now to the integral $\iint Q_pQ_qQ_r dS$, let us expand each of the Q's according to LAPLACE'S Formula. We thus find for the general term

$$\frac{2^{4(\alpha+\beta)-3}p!\,p!\,q!\,q!\,r!\,r!}{p+\alpha!\,p-\alpha!\,q+\beta!\,q-\beta!\,r+\alpha+\beta!\,r-\alpha-\beta!}$$

$$\times\iint\{(p,\alpha)_a(p,\alpha)+[p,\alpha]_a[p,\alpha]\}\{(q,\beta)_b(q,\beta)+q,\beta]_b[q,\beta]\}\{(r,\alpha+\beta)_c(r,\alpha+\beta)$$

$$+[r,\alpha+\beta]_c[r,\alpha+\beta]\}dS$$

Now, by the formula of § 10, when we come to equate the two results just found, we shall get

$$\frac{2^{2(\alpha+\beta)-1}p!\,q!\,r!}{p+\alpha!\,q+\beta!\,r-\alpha-\beta!}\iint(p,\alpha)(q,\beta)(r,\alpha+\beta)dS=\frac{4\pi R^2}{2i+1!}\frac{i!\,2\lambda!\,2\mu!\,2\nu!}{\lambda!\,\mu!\,\nu!}W$$

or finally

$$\iint(p,\alpha)(q,\beta)(r,\alpha+\beta)dS=\frac{4\pi R^2}{2i+1!}\frac{i!\,2\lambda!\,2\mu!\,2\nu!\,p+\alpha!\,q+\beta!\,r-\alpha-\beta!}{\lambda!\,\mu!\,\nu!\,p!\,q!\,r!}W. \quad . \quad (29)$$

where W is defined above (28) as the coefficient of a term in a certain product, or as a series whose terms depend upon a single parameter.

If we denote by U the value of the expression on the right hand of the equation just found, we have further

$$\iint(p,\alpha)(q,\beta)[r,\alpha+\beta]d\mathrm{S}=0$$

$$\iint(p,\alpha)[q,\beta][r,\alpha+\beta]d\mathrm{S}=-\mathrm{U}$$

&c.

$$\iint \mathrm{P}_p\mathrm{P}_q\mathrm{P}_r\mathrm{P}_s d\mathrm{S}$$

---

§ 14. We may remark in regard to the case which we have now to examine, that since $\mathrm{P}_r\mathrm{P}_s$ may by the foregoing results be expanded in a series of the form

$$\mathrm{A}_0\mathrm{P}_{r-s}+\mathrm{A}_1\mathrm{P}_{r-s+2}+\ \ldots\ +\mathrm{A}_m\mathrm{P}_{r-s+2m}+\ \ldots\ +\mathrm{A}_s\mathrm{P}_{r+s}$$

where the value of A_m determined by (26) is

$$\frac{2r-2s+4m+1}{2r+2m+1}\ \frac{\underline{r+m}!\ \underline{r+m}!\ \underline{2m}!\ \underline{2r-2s+2m}!\ \underline{2s-2m}!}{\underline{2r+2m}!\ m!\ m!\ \underline{r-s+m}!\ \underline{r-s+m}!\ \underline{s-m}!\ \underline{s-m}!}$$

the above integral is reducible to the computation of a series of s terms of the form

$$\mathrm{A}_m\iint \mathrm{P}_p\mathrm{P}_q\mathrm{P}_{r-s+2m}d\mathrm{S}$$

The last integral is to be found by writing in (26)

$$2\lambda=-p+q+r-s+2m$$

$$2\mu=p-q+r-s+2m$$

$$2\nu=p+q-r+s-2m$$

We are thus led to a complete though practically laborious evaluation of $\iint \mathrm{P}_p\mathrm{P}_q\mathrm{P}_r\mathrm{P}_s d\mathrm{S}$. It will be observed that the result is zero unless $p+q+r+s$ be an even number.

With the view of finding out how far the method of this paper is applicable in this case, and what difficulties stand in the way of its general application, we will now briefly apply it. It will be convenient to expand the operator in a somewhat different manner to that pursued in the case of the product of three harmonics. The operator was then expanded in a form which would render it useful for application in the case of the tesseral and sectorial system. If, however, the poles of the harmonic are all in the axis of z, a much simpler mode of expansion may be adopted.

When the product of four harmonics is under consideration we have to discuss an operator of the form

$$\left\{\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\frac{d}{dx_3}+\frac{d}{dx_4}\right)^2+\left(\frac{d}{dy_1}+\frac{d}{dy_2}+\frac{d}{dy_3}+\frac{d}{dy_4}\right)^2+\left(\frac{d}{dz_1}+\frac{d}{dz_2}+\frac{d}{dz_3}+\frac{d}{dz_4}\right)^2\right\}^i$$

or, what is the same thing,

$$\left\{\left(\frac{d}{dz_1}+\frac{d}{dz_2}+\frac{d}{dz_3}+\frac{d}{dz_4}\right)^2+4\left(\frac{d}{d\xi_1}+\frac{d}{d\xi_2}+\frac{d}{d\xi_3}+\frac{d}{d\xi_4}\right)\left(\frac{d}{d\eta_1}+\frac{d}{d\eta_2}+\frac{d}{d\eta_3}+\frac{d}{d\eta_4}\right)\right\}^i$$

If this operate upon the reciprocal of $\rho_1\rho_2\rho_3\rho_4$, where these distances are measured from four points A, B, C, D on the axis of z, we have four relations of the form

$$\frac{d^2}{dz}+4\frac{d^2}{d\xi d\eta}=0$$

Further, on picking out the general term, we observe that, to be effectually operative, it must consist of products of the form

$$\frac{d^l}{dz}\frac{d^m}{d\xi}\frac{d^m}{d\eta}$$

and this form of operator may, by the foregoing relation, be expressed in terms of $\frac{d}{dz}$ only.

It is obvious, however, that the sum of the terms so modified can be found directly from the part of the expansion not containing k's of

$$\left\{\left(\frac{d}{dz_1}+\frac{d}{dz_2}+\frac{d}{dz_3}+\frac{d}{dz_4}\right)^2-(k_1+k_2+k_3+k_4)\left(\frac{1}{k_1}\frac{d^2}{dz_1}+\frac{1}{k_2}\frac{d^2}{dz_2}+\frac{1}{k_3}\frac{d^2}{dz_3}+\frac{1}{k_4}\frac{d^2}{dz_4}\right)\right\}^i$$

or, what is the same thing, from the part of the expansion not containing k's of

$$\left\{\left(\frac{d}{dz_1}+\frac{d}{dz_2}+\frac{d}{dz_3}+\frac{d}{dz_4}\right)^2-\left(k_1\frac{d}{dz_1}+k_2\frac{d}{dz_2}+k_3\frac{d}{dz_3}+k_4\frac{d}{dz_4}\right)\left(\frac{1}{k_1}\frac{d}{dz_1}+\frac{1}{k_2}\frac{d}{dz_2}+\frac{1}{k_3}\frac{d}{dz_3}+\frac{1}{k_4}\frac{d}{dz_4}\right)\right\}^i$$

that is, of

$$(-1)^i\left\{\left(\sqrt{\frac{k_1}{k_2}}-\sqrt{\frac{k_2}{k_1}}\right)^2\frac{d}{dz_1}\frac{d}{dz_2}+5\text{ similar terms}\right\}^i$$

If we expand this the general term will be

$$(-1)^i\frac{i!}{\lambda!\,\mu!\,\nu!\,l!\,m!\,n!}\times(k_2-k_3)^{2\lambda}(k_3-k_1)^{2\mu}(k_1-k_2)^{2\nu}(k_4-k_1)^{2l}(k_4-k_2)^{2m}(k_4-k_3)^{2n}$$
$$\times\left(\frac{1}{k_1}\frac{d}{dz_1}\right)^p\left(\frac{1}{k_2}\frac{d}{dz_2}\right)^q\left(\frac{1}{k_3}\frac{d}{dz_3}\right)^r\left(\frac{1}{k_3}\frac{d}{dz_4}\right)^s$$

where

$$\left.\begin{aligned}\mu+\nu+l&=p\\ \nu+\lambda+m&=q\\ \lambda+\mu+n&=r\\ l+m+n&=s\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(30)$$

The quantities λ, μ, ν, l, m, n are therefore indeterminate, and there will be a series of terms having the required indices.

The coefficient of the operator given above may be put into another form. Let us consider the expansion of

$$\{a(k_2-k_3)+b(k_3-k_1)+c(k_1-k_2)+d(k_4-k_1)+e(k_4-k_2)+f(k_4-k_3)\}^{2i}$$

first in powers of $a, b, c, \ldots$ and then in powers of $k_1, k_2, k_3, \ldots$ We then see that the coefficient of $k_1^p\, k_2^q\, k_3^r\, k_4^s$ in $(k_2-k_3)^{2\lambda} \ldots (k_4-k_3)^{2n}$ is equal to

$$\frac{2\lambda!\; 2\mu!\; 2\nu!\; 2l!\; 2m!\; 2n!}{p!\; q!\; r!\; s!}\mathrm{W}$$

where W is the coefficient of $a^{2\lambda}\, b^{2\mu}\, c^{2\nu}\, d^{2l}\, e^{2m}\, f^{2n}$ in

$$(-b+c-d)^p(-c+a-e)^q(-a+b-f)^r(d+e+f)^s$$

We have, finally,

$$\iint \mathrm{P}_p\mathrm{P}_q\mathrm{P}_r\mathrm{P}_s d\mathrm{S}=(-1)^i\frac{4\pi \mathrm{R}^2 i!}{2i+1!}\,\Sigma\,\frac{2\lambda!\; 2\mu!\; 2\nu!\; 2l!\; 2m!\; 2n!}{\lambda!\; \mu!\; \nu!\; l!\; m!\; n!}\mathrm{W}$$

where Σ denotes the sum of all such values of W multiplied by their respective coefficients corresponding to values of λ, μ, ν, l, m, n determined by equations (30).

This investigation serves to exhibit the peculiar practical difficulties which beset the problem of integrating products of harmonics over the sphere of reference, if more than three harmonics be considered.

POTENTIALS OF ELLIPSOIDS.

§ 15. Let p be the perpendicular upon the tangent plane at any point of an ellipsoid, and let us consider the integral

$$\iint e^{\alpha x+\beta y+\gamma z}p\,d\mathrm{S}$$

taken over the surface.

By the theory of corresponding points this integral may clearly be thrown into the form of an integral taken over the surface of a sphere of radius R, viz.: it is

$$\frac{abc}{\mathrm{R}^2}\iint e^{\frac{a\alpha x'+b\beta y'+c\gamma z'}{\mathrm{R}}}d\mathrm{S}',$$

the value of which, by what we have already shown in (3) is

$$2\pi abc\frac{e^{\nabla}-e^{-\nabla}}{\nabla}$$

where

$$\nabla^2=a^2\alpha^2+b^2\beta^2+c^2\gamma^2.$$

Now let V be any function of x, y, z, which can be expanded by TAYLOR'S Theorem; then

$$\iint \mathrm{V}p d\mathrm{S}$$

taken over the surface of the ellipsoid is

$$2\pi abc\frac{e^{\nabla}-e^{-\nabla}}{\nabla}\mathrm{V}$$

or, in series,

$$4\pi abc\left(1+\frac{\nabla^2}{3!}+\ldots+\frac{\nabla^{2i}}{2i+1!}+\ldots\right)\mathrm{V}$$

where

$$\nabla^2=a^2\frac{d^2}{dx}+b^2\frac{d}{dy^2}+c^2\frac{d^2}{dz},$$

and after the differentiations are performed, x, y, z are to be put equal to zero.

§ 16. If V satisfies LAPLACE'S equation, then the operator ∇^2 will remain unaltered if for a^2, b^2, c^2 we substitute $a^2+\epsilon$, $b^2+\epsilon$, $c^2+\epsilon$. It follows, therefore, that the average value of V over any ellipsoid as measured by

$$\frac{\iint\frac{\mathrm{V}pd\mathrm{S}}{3}}{\text{volume of ellipsoid}} \quad \ldots\ldots\ldots\ldots \quad (31)$$

is the same for that ellipsoid as for any ellipsoid confocal with it.

This theorem,* which may be regarded as the anologue for ellipsoids of the corresponding theorem given by GAUSS for spheres, is due to Professor CHARLES NIVEN, who also showed that, if V be due to attracting matter E inside the surface of the ellipsoid, then the expression (31) becomes

$$\frac{\mathrm{E}}{2}\int_0^\infty\frac{d\psi}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}}$$

§ 17. As a particular case, let $\mathrm{V}=\frac{1}{\mathrm{QP}}$ where Q is any point outside of the ellipsoid. Then

$$\iint\frac{pd\mathrm{S}}{\mathrm{QP}}$$

is the potential at Q due to matter of density p coated over the ellipsoid. The quantity of matter is $4\pi abc$ and the potential due to it at Q, according to a well known result is

$$2\pi abc\int_\epsilon^\infty\frac{d\psi}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}}$$

or

$$2\pi abc\alpha$$

where α is the ellipsoidal coordinate of Q.

* Mathematical Tripos Solutions for 1878.

If P′ be any point inside the ellipsoid we then have ultimately when P′ coincides with the origin

$$\alpha=\frac{e^{\nabla}-e^{-\nabla}}{\nabla}\frac{1}{\text{QP}'}$$

But, if f, g, h be the coordinates of Q,

$$\text{QP}'^2=(f-x)^2+(g-y)^2+(h-z)^2$$

Hence

$$\frac{d}{dx}\frac{1}{\text{QP}'}=-\frac{d}{df}\frac{1}{\text{QP}'}$$

$$\frac{d^2}{dx}\frac{1}{\text{QP}'}=\frac{d^2}{df}\frac{1}{\text{QP}'}\qquad\text{\&c.}$$

We may therefore take ∇^2 to stand for

$$a^2\frac{d^2}{df}+b^2\frac{d^2}{dg}+c^2\frac{d^2}{dh}$$

and in that case we may put x, y, z zero before differentiating. If then OQ be denoted by r, we have

$$\alpha=\frac{e^{\nabla}-e^{-\nabla}}{\nabla}\frac{1}{r}\quad\ldots\ldots\ldots\ldots\quad(32)$$

or, in series,

$$2\left(1+\frac{1}{3!}\nabla^2+\ldots+\frac{1}{2i+1!}\nabla^{2i}+\ldots\right)\frac{1}{r}\quad\ldots\ldots\quad(33)$$

If we multiply α by $2\pi\rho abc\delta\theta$ we have the potential of an infinitely thin shell bounded by two similar and similarly situated ellipsoids. When a shell is hereafter referred to, this kind of shell will be meant, unless otherwise specified.

§ 18. Since the whole ellipsoid may be divided up into shells, the potential of the ellipsoid at Q is

$$\iiint\rho\frac{pd\text{S}d\theta}{\text{QP}\theta}\quad\ldots\ldots\ldots\ldots\quad(34)$$

where ρ is the density at any point in the shell whose semi-axes are $a\theta$, $b\theta$, $c\theta$. It will be observed that, since $p\delta\theta=\theta\delta p$, $\frac{p\delta\text{S}\delta\theta}{\theta}$ is an element of volume, so that (34) may be written

$$\iiint\frac{\rho dxdydz}{\text{QP}}$$

If we take the series (33) we find for the potential of the ellipsoid at Q the series

$$\int_0^1 4\pi\rho abc\theta^2\left(1+\frac{\theta^2}{3!}\nabla^2+\ldots+\frac{\theta^{2i}}{2i+1!}\nabla^{2i}+\text{\&c.}\right)\frac{1}{r}d\theta\quad\ldots\ldots\quad(35)$$

This can be integrated if ρ be a function of θ.

In the particular case where ρ is constant and the mass is denoted by M, we find

$$\mathrm{M}\left(1+\frac{3}{5}\frac{1}{3!}\nabla^2+\ldots\ldots+\frac{3}{2i+3}\frac{\nabla^{2i}}{2i+1!}+\ldots\right)\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

§ 19. The result just formed exhibits in a very simple manner MACLAURIN'S Theorem, that the attractions of confocal ellipsoids at external points are proportional to their masses. For, as has been already pointed out, the operator ∇^2 is unaltered by the addition of $k\left(\frac{d^2}{df}+\frac{d^2}{dg}+\frac{d^2}{dh}\right)$.

The series (35) also shows in what direction MACLAURIN'S Theorem may be generalised.

(i.) Let $\rho=f(\theta)$. We then see that if there be two ellipsoids which would be confocal if they were coaxal, and if the matter in them were arranged in layers similar to their bounding surfaces according to any specified law depending on the parameter of the layer, then the attraction of such ellipsoids at external points whose coordinates referred to the axes of the ellipsoids are equal, are proportional to the masses of the ellipsoids.

(ii.) If we multiply the value of α given in (33) by

$$2\pi\sqrt{(a^2-\phi)(b^2-\phi)(c^2-\phi)}\mathrm{F}(\phi)d\phi$$

and integrate between the values ϕ_1 and ϕ_2, we find on the right hand side

$$4\pi\int_{\phi_2}^{\phi_1}\sqrt{(a^2-\phi)(b^2-\phi)(c^2-\phi)}\mathrm{F}(\phi)d\phi\times\left(1+\frac{\nabla^2}{3!}+\ldots+\frac{\nabla^{2i}}{2i+1!}+\ldots\right)\frac{1}{r} \quad . \quad (37)$$

in which the operator is independent of ϕ.

On the left hand side we have

$$\int_{\phi_2}^{\phi_1}2\pi\sqrt{(a^2-\phi)(b^2-\phi)(c^2-\phi)}\alpha\mathrm{F}(\phi)d\phi$$

$$=\int_{\phi_2}^{\phi_1}\iint\frac{\mathrm{F}(\phi)p\,d\mathrm{S}}{\mathrm{QP}}d\phi$$

$$=\iiint\frac{2\mathrm{F}(\phi)p^2}{\mathrm{QP}}dx\,dy\,dz$$

where p is the perpendicular from the centre on the tangent plane at the point x, y, z to the confocal, passing through the point of parameter ϕ.

Hence, if the matter between two confocal ellipsoids be affected with a density varying as $\mathrm{F}(\phi)p^2$, the equipotential surfaces will be confocal ellipsoids, as may be seen from the expression (37), which also gives the ratios of the attractions due to different shells.

(iii.) The theorem is true for any value of V satisfying LAPLACE'S equation.

§ 20. If we put $a=b$ then ∇^2 becomes $-(a^2-c^2)\frac{d^2}{dz}$, and the potential of a solid ellipsoid of revolution of uniform density becomes

$$\frac{M}{r}\left(1-\frac{3P_2}{3\cdot5}\frac{a^2-c^2}{r^2}+\frac{3P_4}{5\cdot7}\left(\frac{a^2-c^2}{r^2}\right)^2-\frac{3P_6}{7\cdot9}\left(\frac{a^2-c^2}{r^2}\right)^3+\ldots\right) \quad . \quad . \quad . \quad . \quad (38)$$

§ 21. The expansion involving surface spherical harmonics of the potential of a solid ellipsoid of uniform density is easily derived from (36).

In virtue of LAPLACE'S equation, the operator ∇^2 becomes

$$(a^2-b^2)\frac{d^2}{dx}-(b^2-c^2)\frac{d^2}{dz}$$

which we will abbreviate to

$$f^2\frac{d^2}{dx}-g^2\frac{d^2}{dz}$$

Now by Theorem i.

$$\left(f^2\frac{d^2}{dx}-g^2\frac{d^2}{dz}\right)^i\frac{1}{r}=(-1)^i\frac{1}{r^{2i+1}}\left(g^2\frac{d^2}{dz}-f^2\frac{d^2}{dx}\right)^i r^{2i}Q_{2i}$$

Since there is here no operator in regard to y we may put $y=0$ in the expression for $Q_{2i}r^{2i}$ before differentiating. In that case $\xi=\eta=x$, and the expansion will take a comparatively simple form.

We take, according to custom, the axis of z for the axes of the zonal and tesseral-sectorial system. The expansion of $Q_{2i}r^{2i}$ then becomes, by LAPLACE'S Formula

$$P'_{2i}r^{2i}P_{2i}+\ldots\frac{2^{4\sigma-1}\,2i!\,2i!}{2i+2\sigma!\,2i-2\sigma!}(2i,\,2\sigma)'r^{2i}(2i,\,2\sigma)+\ldots$$

$$=P'_{2i}\left(z^{2i}-\frac{2i(2i-1)}{4}z^{2i-2}x^2+\ldots\right)+\ldots+\ldots$$

$$+\frac{2i!}{2\sigma!\,2i-2\sigma!}(2i,\,2\sigma)'\left(z^{2i-2\sigma}x^{2\sigma}-\frac{(2i-2\sigma)(2i-2\sigma-1)}{4(2\sigma+1)}z^{2i-2\sigma-2}x^{2\sigma+2}+\ldots\right)+\text{\&c.}$$

The general term of the expansion of the potential of an ellipsoid may therefore be expressed thus :—

$$(-1)^i\frac{3}{(2i+3)(2i+1)}\frac{1}{r^{2i+1}}(AP'_{2i}+\ldots+B_{2\sigma}(2i,\,2\sigma)'+\ldots)$$

where

$$A=g^{2i}-\frac{i}{2}g^{2i-2}f^2+\frac{i(i-1)}{1.2}\frac{3}{2}\frac{1}{2^2}g^{2i-4}f^4-\text{\&c.}$$

$$B_{2\sigma}=(-1)^\sigma\frac{i!}{\sigma!\,i-\sigma!}\left(g^{2i-2\sigma}f^{2\sigma}-\frac{i-\sigma}{2}g^{2i-2\sigma-2}f^{2\sigma+2}\right.$$

$$\left.+\frac{(i-\sigma)(i-\sigma-1)}{1.2}\frac{2\sigma+3}{2\sigma+2}\frac{1}{2^2}g^{2i-2\sigma-4}f^{2\sigma+4}-\text{\&c.}\right) \quad . \quad . \quad . \quad . \quad (39)$$

§ 22. It may be observed that the symbolical form of the operator in the case of the potential of a shell leads to a symbolical form for the potential of the ellipsoid, viz. :—

$$\frac{3M}{2}\frac{\nabla(e^{\nabla}+e^{-\nabla})-e^{\nabla}+e^{-\nabla}}{\nabla^3}\frac{1}{r}$$

Now let there be a second solid ellipsoid of mass M′ and of semi-axes a', b', c', and let it be placed in a position where the coordinates of its centre referred to the axes of the first ellipsoid are f, g, h, and its axes are inclined at direction cosines $(l_1\ m_1\ n_1)$, $(l_2\ m_2\ n_2)$, $(l_3\ m_3\ n_3)$. Then if we put

$$\nabla_1^2=a'^2\frac{d^2}{dx}+b'^2\frac{d^2}{dy}+c'^2\frac{d^2}{dz}$$

where

$$\frac{d}{dx}=l_1\frac{d}{df}+m_1\frac{d}{dg}+n_1\frac{d}{dh}$$

$$\frac{d}{dy}=l_2\frac{d}{df}+m_2\frac{d}{dg}+n_2\frac{d}{dh}$$

$$\frac{d}{dz}=l_3\frac{d}{df}+m_3\frac{d}{dg}+n_3\frac{d}{dh}$$

a double application of the reasoning of §§ 16–18 leads us to the conclusion that the exhaustion of the potential energy of the two ellipsoids due to mutual action is

$$\frac{9}{4}MM'\frac{\nabla(e^{\nabla}+e^{-\nabla})-e^{\nabla}+e^{-\nabla}}{\nabla^3}\frac{\nabla_1(e^{\nabla_1}+e^{-\nabla_1})-e^{\nabla_1}+e^{-\nabla_1}}{\nabla_1^3}\frac{1}{r} \quad . \quad . \quad . \quad . \quad (40)$$

where r is the distance between their centres.

If we turn the second ellipsoid through an angle $\delta\theta$ about the line (λ, μ, ν) passing through its centre, we have

$$\left.\begin{aligned}\delta l_1&=(\nu m_1-\mu n_1)\delta\theta\\ \delta m_1&=(\lambda n_1-\nu l_1)\delta\theta\\ \delta n_1&=(\mu l_1-\lambda m_1)\delta\theta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

with similar increments for the other direction cosines. Hence the expansion of (40) in harmonics leads to an approximate determination of the forces and couples representing the mutual action between the two ellipsoids.

It is obvious that a similar investigation will apply to the determination of the forces and couples between two magnets in the form of ellipsoids uniformly magnetised.

It may be interesting to notice that the foregoing method of expansion (36) shows that for points at a considerable distance the potential due to a solid ellipsoid is the same as if its mass were distributed as follows :—Two-fifths at the centre and one-tenth at the extremities of each of the axes.

General Remarks on Ellipsoidal Surface and Volume Integrals.

§ 23. We have found for the ellipsoid that

$$\iint \mathrm{V}pd\mathrm{S}=4\pi abc\left(1+\frac{\nabla^2}{3!}+\ldots\right)\mathrm{V}$$

$$\iiint \mathrm{V}dxdydz=4\pi abc\left(\frac{1}{3}+\frac{1}{5}\frac{\nabla^2}{3!}+\ldots\right)\mathrm{V}$$

where V is any function of x, y, z whose differential coefficients are finite at the origin. It is obvious, therefore, that these two results lead to an infinite variety of definite integrals. For example, let us use the second result in finding the value of $\iiint x^{2l}y^{2m}z^{2n}dxdydz$, which is an integral evaluated by LAGRANGE in determining the potential due to an ellipsoid (TODHUNTER'S 'History of the Theory of Attractions,' vol. ii, p. 158).

Putting $i=l+m+n$, we find at once

$$\iiint x^{2l}y^{2m}z^{2n}dxdydz=\frac{4\pi abc\ i!\ a^{2l}b^{2m}c^{2n}}{(2i+3)(2i+1)!\ l!\ m!\ n!}\cdot\frac{d^{2l}}{dx}\frac{d^{2m}}{dy}\frac{d^{2n}}{dz}x^{2l}y^{2m}z^{2n}$$

$$=4\pi a^{2l+1}b^{2m+1}c^{2n+1}\frac{1.3.5\ldots(2l-1)1.3.5\ldots(2m-1)1.3.5\ldots(2n-1)}{1.3.5\ldots(2l+2m+2n+3)}$$

If we suppose the function V such that it satisfies the equation

$$a^2\frac{d^2\mathrm{V}}{dx^2}+b^2\frac{d^2\mathrm{V}}{dy^2}+c^2\frac{d^2\mathrm{V}}{dz^2}=0 \quad \ldots\ldots\ldots \quad (42)$$

then the surface volume integrals are the simplest possible in regard to results, but it will depend upon some other condition attached to V what class of function we shall have succeeded in integrating. For example, if V also satisfies the condition

$$x\frac{d\mathrm{V}}{dx}+y\frac{d\mathrm{V}}{dy}+z\frac{d\mathrm{V}}{dz}=n\mathrm{V}$$

we should be led to a class of results similar to those obtained in the case of spherical harmonics for the sphere, and, in fact, derivable from these by the theory of corresponding points. As another example, let us suppose that V, besides satisfying equation (42), also satisfies LAPLACE'S equation; then one solution for V will be of the form

$$\mathrm{F}(\sqrt{b^2-c^2}x+\sqrt{c^2-a^2}y+\sqrt{a^2-b^2}z)$$

We thus get

$$\left.\begin{aligned}\iint \mathrm{F}pd\mathrm{S}\quad &=4\pi abc\mathrm{F}(0)\\ \iiint \mathrm{F}dxdydz&=\frac{4\pi abc}{3}\mathrm{F}(0)\end{aligned}\right\}\quad \ldots\ldots\ldots \quad (43)$$

As a limiting case, let the ellipsoid wear down to the focal ellipse, and for the sake of brevity let the semi-axes of that ellipse be denoted by f and g. We find

$$\left.\begin{aligned}\iint\frac{\mathrm{F}(gx+jfy)}{\sqrt{1-\frac{x^2}{f^2}-\frac{y^2}{g^2}}}dxdy&=2\pi fg\mathrm{F}(0)\\ \iint\sqrt{1-\frac{x^2}{f^2}-\frac{y^2}{g^2}}\,\mathrm{F}(gx+jfy)dxdy&=\frac{2\pi fg}{3}\mathrm{F}(0)\end{aligned}\right\}\quad\ldots\ldots\quad(44)$$

These results may be further simplified if the ellipse becomes a circle.

The Potential due to an Ellipse of uniform density.

§ 24. The integral

$$\int e^{\alpha x+\beta y}pds$$

taken round the perimeter of an ellipse of semi-axes a, b, is derivable from the corresponding case of the circle of radius R by the theory of corresponding points, and the result is

$$2\pi ab\sum_0^\infty\frac{1}{2^{2i}\,i!\,i!}(a^2\alpha^2+b^2\beta^2)^i\quad\ldots\ldots\ldots\quad(45)$$

If we write herein $a\theta$, $b\theta$, and recollect that $p\delta\theta=\theta\delta p$, then

$$\iint e^{\alpha x+\beta y}dxdy$$

taken over the area of the ellipse is equal to

$$\iint e^{\alpha x+\beta y}\frac{p}{\theta}dsd\theta$$

the limits of θ being from 0 to 1. The result of integrating this is

$$2\pi ab\sum_0^\infty\frac{(a^2\alpha^2+b^2\beta^2)^i}{2^{2i+1}\,i!\,\overline{i+1}!}\quad\ldots\ldots\ldots\quad(46)$$

Similar reasoning would give us

$$\iint f\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)e^{\alpha x+\beta y}dxdy$$

Confining our attention, however, to the result just found (46), we see that the integration over the area of the ellipse of any function V of x, y, having finite differential coefficients at the origin, leads to

$$\iint \mathrm{V}\,dxdy = 2\pi ab \sum_0^\infty \frac{\left(a^2\frac{d^2}{dx}+b^2\frac{d^2}{dy}\right)^i}{2^{2i+1}\, i!\, \overline{i+1}!}\mathrm{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where, after the differentiations are performed, x, y, z are to be put equal to zero.

If we suppose the ellipse coated with matter of unit density we thus get for the potential at any outside point (f, g, h), sufficiently distant,

$$\iint \frac{dxdy}{\sqrt{(f-x)^2+(g-y)^2+h^2}} = 2\pi ab \sum_0^\infty \frac{\left(a^2\frac{d^2}{dx}+b^2\frac{d^2}{dy}\right)^i}{2^{2i+1}\, i!\, \overline{i+1}!}\,\frac{1}{r}$$

$$= 2\pi ab \sum_0^\infty \frac{\left(a^2\frac{d^2}{df}+b^2\frac{d^2}{dg}\right)^i}{2^{2i+1}\, i!\, \overline{i+1}!}\,\frac{1}{\sqrt{f^2+g^2+h^2}}$$

$$= 2\pi ab \sum_0^\infty \frac{(-1)^i 2i!}{2^{2i+1}\, i!\, \overline{i+1}!}\,\frac{1}{r^{i+1}}\left(\mathrm{A}\mathrm{P}'_{2i} + \ldots + \mathrm{B}_{2\sigma}(2i, 2\sigma) + \ldots\right) \quad . \quad . \quad (48)$$

Where A, . . . $\mathrm{B}_{2\sigma}$ . . . are the same as in § 21, provided we write $f^2=a^2-b^2$ and $g^2=b^2$ in the expansion of that article. We must have $r>b$ and $>\sqrt{a^2-b^2}$, if the above expansion is convergent.

If $a=b$ or $f=0$ the expansion just found reduces to the well known expansion for a circular plate of uniform thickness, given in THOMSON and TAIT, p. 406.

(Added September 23, 1879.)

§ 25. The series given above for the potential of an ellipse may be thrown into the form of an integral. Writing the result for the elliptic ring in the form

$$2\pi ab \sum_0^\infty \frac{\left\{(a^2-b^2)\frac{d^2}{df}-b^2\frac{d^2}{dh}\right\}^i}{2^{2i}\, i!\, i!}\,\frac{1}{r}$$

we see that this series is the part of the expansion not involving powers of k or its reciprocal of

$$2\pi ab e^{\frac{kj}{2}\left(b\frac{d}{dh}+c\frac{d}{df}\right)}\, e^{\frac{j}{2k}\left(b\frac{d}{dh}-c\frac{d}{df}\right)}\frac{1}{r}$$

where $c^2=a^2-b^2$. That is, the series is equal to

$$ab\int_0^{2\pi} e^{jb\cos\psi\frac{d}{dh}-c\sin\psi\frac{d}{df}}\,\frac{1}{r}\,d\psi.$$

Hence writing herein $a\theta$, $b\theta$, multiplying by $\frac{d\theta}{\theta}$ and integrating, we find for the potential at (f, g, h) the expression

$$ab\int_0^{2\pi}\int_0^1 e^{\left(jb\cos\psi\frac{d}{dh}-c\sin\psi\frac{d}{df}\right)\theta}\frac{1}{r}\theta d\theta d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

The expansion of the exponential leads of course to the series (48) given above, and it is easy to find thereby the expressions for the action of an elliptic current on a magnet placed at any distance from a coil outside a certain boundary. In like manner, subject to a similar restriction, we might find, as in the case of two ellipsoids, the action of one elliptic current on another. Within the boundary referred to, the expressions we have obtained are no longer convergent, and in the case of the elliptic current the potential in the neighbourhood of the centre of the circuit must be found by an independent process.

The case of the circle, however, admits of a simple and complete determination.

Potential of Electric Currents in Circular Circuits.

§ 26. Reverting to the integral (a) let us put $c=0$; then the potential at (f, g, h) due to a circle of unit density is given by

$$V=a^2\int_0^{2\pi}\int_0^1 e^{ja\theta\cos\psi\frac{d}{dh}}\frac{1}{r}\theta d\theta d\psi$$

To integrate this in regard to θ we observe first that

$$\int_0^1\theta e^{u\theta}d\theta=\frac{e^u}{u}-\frac{e^u}{u^2}+\frac{1}{u^2}$$

Let us next consider the integral

$$\int_0^{2\pi}e^{k\cos\psi}d\psi, \text{ or } \int_0^{2\pi}\cos\psi\frac{e^{k\cos\psi}}{\cos\psi}d\psi$$

Integrating by parts we find

$$\left[e^{k\cos\psi}\tan\psi\right]_0^{2\pi}+\int_0^{2\pi}\frac{k\sin^2\psi}{\cos\psi}e^{k\cos\psi}d\psi-\int_0^{2\pi}\frac{\sin^2\psi}{\cos^2\psi}e^{k\cos\psi}d\psi$$

The first term becomes infinite when $\psi=\frac{\pi}{2}$ and when $\psi=\frac{3\pi}{2}$; we may replace it by $\int_0^{2\pi}\sec^2\psi d\psi$. We thus find

$$\int_0^{2\pi}\left(\frac{e^{k\cos\psi}}{k\cos\psi}-\frac{e^{k\cos\psi}}{k^2\cos^2\psi}+\frac{1}{k^2\cos^2\psi}\right)d\psi=\frac{1}{k}\int_0^{2\pi}\cos\psi e^{k\cos\psi}d\psi=\int_0^{2\pi}\sin^2\psi e^{k\cos\psi}d\psi$$

Combining these results, we find for the potential at (f, g, h) due to a circular plate of unit density the expression

$$a^2\int_0^{2\pi}\sin^2\psi e^{ja\cos\psi\frac{d}{dh}}\frac{1}{r}d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

If we move the origin to a point at distance v on the negative side of the axis of z, we find

$$2a^2\int_0^{\pi}e^{-(ja\cos\psi+v)\frac{d}{dh}}\frac{1}{r}d\psi$$

in which form the expansion of $-\frac{dV}{dh}$ readily gives CLERK MAXWELL'S expansion for the potential of unit current in the circle ('Electricity and Magnetism,' vol. ii., p. 301 (6′)).

The result expressed by (b) may be written

$$V=a^2\int_0^{2\pi}\frac{\sin^2\psi}{\sqrt{f^2+g^2+(h+aj\cos\psi)^2}}d\psi$$

Hence the potential due to unit current in the circle is

$$-a^2\int_0^{2\pi}\sin^2\psi\frac{d}{dh}\frac{1}{\sqrt{f^2+g^2+(h+aj\cos\psi)^2}}d\psi$$

$$=aj\int_0^{2\pi}\sin\psi\frac{d}{d\psi}\frac{1}{\sqrt{f^2+g^2+(h+aj\cos\psi)^2}}d\psi$$

$$=\int_0^{2\pi}\frac{aj\cos\psi}{\sqrt{f^2+g^2+(h+aj\cos\psi)^2}}d\psi$$

$$=2\int_0^{\pi}\frac{aj\cos\psi}{\sqrt{f^2+g^2+(h+aj\cos\psi)^2}}d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

This result shows that the potential of a circular current is the same as that of an imaginary bar in the axis of z joining the points whose distances from the origin are aj and $-aj$, the density at a point ψ being $-2\cot\psi$. Now the integral (c) is true at all points whose distances from the origin are greater than a. We can, however, determine the corresponding result for points within the radius a by the ordinary theory of inversion. We have seen that for points outside the radius a the potential is the same as that of a bar joining two imaginary points. If z be the distance of any point of this bar from the origin and z' the corresponding inverted distance, we have

$$z' = \frac{a^2}{z} = -aj \sec\psi$$

$$\therefore dz' = -aj \sec\psi \tan\psi d\psi$$

and the corresponding density is $-2\cot\psi \dfrac{a}{aj\cos\psi}$

Hence the potential at all points inside the radius a is

$$2\int_0^\pi \frac{a\sec^2\psi}{\sqrt{f^2+g^2+(h-aj\sec\psi)^2}}d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{d})$$

This result may also be proved from the consideration that the expression for the magnetic force is continuous as we pass from one form of potential to the other.

§ 27. The results (c) and (d) give the mutual potential energy between two circular currents placed in any positions whatever, in which the smaller circle is either wholly outside or wholly inside the radius of the other. For the sake of simplicity we shall suppose that the axes of the two currents intersect. Let the radius of the larger be A and of the smaller a, and let u, v be the distances of their centres from the point of intersection, the axes being inclined at an angle $\cos^{-1}\mu$. Then, if unit current circulates in each, and if the smaller circle lies wholly inside the sphere of radius A containing the larger, the mutual potential energy is given by

$$\mathrm{M} = 4a\mathrm{A}\int_0^\pi\int_0^\pi \frac{j\cos\phi\sec^2\psi}{\sqrt{(u+\mathrm{A}j\sec\psi)+(v+aj\cos\phi)^2-2(u+\mathrm{A}j\sec\psi)(v+aj\cos\phi)\mu}}d\phi d\psi$$

If this series be expanded in harmonics

$$\mathrm{B}_0\mathrm{P}_0 + \mathrm{B}_1\mathrm{P}_1 + \ldots + \mathrm{B}_n\mathrm{P}_n + \ldots$$

we have

$$\mathrm{B}_n = 4a\mathrm{A}\int_0^\pi\int_0^\pi \frac{j\cos\phi\sec^2\psi(v+aj\cos\phi)^n}{(u+\mathrm{A}j\sec\psi)^{n+1}}d\phi d\psi$$

The series in the form of zonal harmonics is given by CLERK MAXWELL ('Electricity and Magnetism,' vol. ii., p. 303).

On comparison with his series, since

$$\int_0^\pi j\cos\phi(\mu+j\nu\cos\phi)^n d\phi = -\frac{\pi}{n+1}\nu\frac{d\mathrm{P}_n}{d\mu}$$

when we take account of the above value of B_n we deduce

$$\int_0^\pi \frac{\sec^2\psi d\psi}{(\mu+\nu j\sec\psi)^{n+1}} = -\frac{\pi}{n}\nu\frac{d\mathrm{P}_n}{d\mu}$$

true for integral values of n, from 1 upwards.

If the smaller circle is wholly outside the sphere of radius A, the value of M given above is inapplicable, and we ought to take a double application of the formula (c).

§ 28. In the important case of two coaxal circular currents, if b be the distance between them, the mutual potential energy is given by

$$M=4aA\int_0^\pi\int_0^\pi\frac{j\cos\phi\sec^2\psi}{b+j(A\sec\psi-a\cos\phi)}d\phi d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

provided the sphere of radius A encloses the smaller circle.

Performing the integration in regard to ψ we find

$$M=4\pi a\int_0^\pi\frac{\cos\phi(a\cos\phi+bj)}{\sqrt{A^2-(a\cos\phi+bj)^2}}d\phi$$

With the view of simplifying this let us put $a\cos\phi+bj=A\cos(x+yj)$, so that $a\cos\phi=A\cos x\cos yj$ and $-bj=A\sin x\sin yj$. Thus x and y are not independent, and by the substitution proposed it can readily be shown that M takes the form

$$8\pi A^2\int\frac{\cos^2 x\sin x}{\sqrt{A^2\cos^4 x-(a^2+A^2+b^2)\cos^2 x+a^2}}dx$$

the limits of integration being determined as follows:

When $\phi=\frac{\pi}{2}$, $x=\frac{\pi}{2}$, and when $\phi=0$, $A^2\cos^4 x-(a^2+A^2+b^2)\cos^2 x+a^2=0$.

The roots of the last equation are

$$p^2=\cos^2\theta_1=\frac{A^2+a^2+b^2+\sqrt{(A^2+a^2+b^2)^2-4a^2A^2}}{2A^2}$$

$$q^2=\cos^2\theta_2=\frac{A^2+a^2+b^2-\sqrt{(A^2+a^2+b^2)-4a^2A^2}}{2A^2}$$

Since p is greater than unity we must take q as the value of $\cos\theta$ when $\phi=0$. Hence the value of M is

$$8\pi A\int_{\cos^{-1}q}^{\frac{\pi}{2}}\frac{\cos^2 x\sin x}{\sqrt{(p^2-\cos^2 x)(q^2-\cos^2 x)}}dx$$

$$=8\pi Aq^2\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta}{\sqrt{p^2-q^2\cos^2\theta}}d\theta$$

$$=8\pi Ap\left\{F\left(\frac{q}{p}\right)-E\left(\frac{q}{p}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

where F and E are complete elliptic integrals whose modulus is $\frac{q}{p}$.

If b be small and if $A-a=c$ be also small, then approximately

$$p=1+\sqrt{\frac{b^2+c^2}{A^2}} \text{ and } q=1-\sqrt{\frac{b^2+c^2}{A^2}}$$

and M becomes

$$4\pi A\left\{\log 8\sqrt{\frac{A^2}{b^2+c^2}}-2\right\}$$

This result is given by CLERK MAXWELL.

§ 29. If the circle of radius a be outside the sphere of radius A passing through the larger circuit, the value of M is

$$4aA\int_0^\pi\int_0^\pi\frac{\cos\phi\cos\psi}{b+j(A\cos\phi+a\cos\psi)}d\phi d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

The forms (e) and (g) obtained for M are interesting from their simplicity, and would seem to be useful in calculations connected with induction coils.